

**INDIVIDUAL LOWER BOUND FOR CALDERON'S  
GENERALIZED LORENTZ NORM ESTIMATES.**

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**ABSTRACT.**

We find the exact values for constants in bilateral Calderon - Stein - Weiss inequalities between tail (Marcinkiewicz) norm and weak Lebesgue (Lorentz) norm.

Possible applications: Functional Analysis (for instance, interpolation of operators), Integral Equations, Probability Theory and Statistics (tail estimations for random variables) etc.

*Key words and phrases:* Tail function, rearrangement invariant norm, random variable, distributions, weight, upper and lower estimates, right inverse function, weak Lebesgue spaces, weak and strong Lorentz, Marcinkiewicz norm and spaces.

*Mathematics Subject Classification (2000):* primary 60G17; secondary 60E07; 60G70.

## **1 Notations. Statement of problem.**

Let  $(X = \{x\}, \mathcal{A}, \mu)$  be measurable space with non-trivial sigma-finite measure  $\mu$ . We will suppose without loss of generality in the case  $\mu(X) < \infty$  that  $\mu(X) = 1$  (the probabilistic case) and denote  $x = \omega$ ,  $\mathbf{P} = \mu$ .

Define as usually for arbitrary measurable function  $f : X \rightarrow R$  its distribution function (more exactly, tail function)

$$T_f(t) = \mu\{x : |f(x)| \geq t\}, \quad t \geq 0,$$

$$\|f\|_p = \left[ \int_X |f(x)|^p \mu(dx) \right]^{1/p}, \quad p \geq 1; \quad L_p = \{f, \|f\|_p < \infty\},$$

and denote by  $f^*(t) = T_f^{-1}(t)$  the right inverse to the tail function  $T_f(t)$  :

$$f^*(t) = \inf\{s : \mu(\{x : |f(x)| > s\}) \leq t\}.$$

The following function  $f^{**} = f^{**}(t)$  play a very important role in the theory of interpolation of operators and harmonic analysis, see [3], [21]:

$$f^{**}(t) \stackrel{\text{def}}{=} t^{-1} \int_0^t f^*(s) ds, \quad t > 0.$$

We will denote the set of all tail functions as  $\{T\}$ ; obviously, the set  $\{T\}$  contains on all the functions  $\{H = H(t), t \geq 0\}$  which are right continuous, monotonically non-increasing with values in the set  $[0, \mu(X)]$ .

Let  $w = w(s), s \geq 0$  be any (measurable) non-increasing numerical function (weight) defined on the set  $s \in (0, \infty)$  such that

$$w(s) = 0 \Leftrightarrow s = 0; \quad \lim_{s \rightarrow \infty} w(s) = \infty. \quad (1.1)$$

The set of all such a functions we will denote as  $V$ ;  $V = \{w\}$ .

Moreover, we introduce the set of all a weight functions  $W = \{w\}$  under another restriction:

$$\forall w \in W \exists T \in \{T\} \Rightarrow w(T(s)) = 1/s. \quad (1.2)$$

Let us introduce the following important functional

$$\gamma(w) = \sup_{t>0} \left[ \frac{w(t)}{t} \int_0^t \frac{du}{w(u)} \right] \quad (1.3)$$

and the following quasi-norms:

$$\|f\|_w^* = \sup_{t>0} [w(t) f^*(t)], \quad (1.4)$$

$$\|f\|_w = \sup_{t>0} [w(t) f^{**}(t)], \quad (1.5)$$

The necessary and sufficient condition for finiteness of the functional  $\gamma(w)$  and following for the normability of the space  $L_w$  see, e.g. in the article [2]; see also [7], [9].

**Remark 1.1.** As long as

$$f^{**}(t) = t^{-1} \sup_{\mu(E) \leq t} \int_E |f(x)| \mu(dx),$$

we can rewrite the expression for  $\|f\|_w$  as follows:

$$\|f\|_w = \sup_{t>0} \left[ (w(t)/t) \cdot \sup_{E: \mu(E) \leq t} \int_E |f(x)| \mu(dx) \right]. \quad (1.6)$$

If the measure  $\mu$  has not atoms, then the expression (1.6) may be rewritten as follows:

$$\|f\|_w = \sup_{E: 0 < \mu(E) < \infty} \left[ \frac{w(\mu(E))}{\mu(E)} \cdot \int_E |f(x)| \mu(dx) \right]. \quad (1.7)$$

It follows from identity (1.6) that  $\|f\|_w$  is rearrangement invariant norm and the space  $L_w = \{f : \|f\|_w < \infty\}$  is (complete) Banach functional rearrangement invariant space with Fatou property. The proof is alike to one in the case  $w(t) = t^{1/p}$ ,  $p \geq 1$ ; see [3], chapters 1,2; [21], chapter 5, section 3.

The norm  $\|f\|_w$  is named Marcinkiewicz's norm, see [10], chapter 2, section 2. More information about considered in this article Marcinkiewicz's and Lorentz (weak Lebesgue) spaces with described applications see, e.g. in [2], [3], chapter 3, section 3; [4], [5], [6], [10], [11], [13], [14], [18], chapter 5, section 5; [20], [21], chapter 5, section 3 etc.

See also many works of L.Maligranda at all [7], [9], [12] etc; M.M.Milman at all [8], [16], [17] etc. which are devoted to the theory of those spaces.

## 2 Main result.

In the article [19] (Theorem 2.1.) has been proved the following estimation, which is some generalization of Calderon - Stein - Weiss bilateral inequality; see also [4] and the classical monographs of E.M. Stein - G.Weiss [21], chapter 5, section 3 and G.O.Okikiolu [18], chapter 5, section 5; if

$$w \in W, \gamma(w) < \infty, \quad (2.1)$$

then

$$1 \cdot \|f\|_w^* \leq \|f\|_w \leq \gamma(w) \cdot \|f\|_w^*, \quad (2.2)$$

and both the coefficients "1" and " $\gamma(w)$ " in (2.2) are the best possible.

**Remark 2.1.** Note that more general version of used in the proving of assertion (2.2) Hardy's inequality is obtained, for example, in [1].

The exactness of the constant "1" follows immediately from the consideration of the case  $w = w_p(s) = w_p(s) := s^{1/p}$ ,  $p > 1$ . Namely, in this case we obtain the classical inequality

$$\|f\|_{w_p}^* \leq \|f\|_{w_p} \leq \frac{p}{p-1} \cdot \|f\|_{w_p}^*,$$

see [21], chapter 5, section 3; note that  $\lim_{p \rightarrow \infty} p/(p-1) = 1$ .

*Notice that the exactness of lower bound in (2.2) is understood over all the set  $W$ , in contradiction to the upper estimation, which is true for every function  $w$ ,  $w \in W$ .*

**The aim of this short report is to prove that the lower coefficient "1" in the left hand-side of bilateral inequality (2.2) is also exact (under some natural conditions) for each function  $w$ ,  $w \in V$ .**

It suffices to consider only the probabilistic case  $\mu(X) = \mathbf{P}(X) = 1$ .

More detail, let us denote

$$\Theta(w) = \inf_{f \neq 0} \left[ \frac{\|f\|_w}{\|f\|_w^*} \right], \quad (2.3)$$

$$G_\kappa(w) = \sup_{t \in (0,1)} [w(t)(1 - t^\kappa)], \quad \kappa = \text{const} \in (0, \infty), \quad (2.4)$$

$$H_\kappa(w) = \sup_{t \in (0,1)} [w(t)(1 - t^\kappa/(\kappa + 1))]. \quad (2.5)$$

$$K_\kappa(w) = \frac{G_\kappa(w)}{H_\kappa(w)}, \quad K(w) = \inf_{\kappa \in (0, \infty)} K_\kappa(w). \quad (2.6)$$

**Theorem 1.**

$$\Theta(w) \leq K(w). \quad (2.7)$$

**Proof.** Let  $\kappa$  be a fix number from the semi-axes  $(0, \infty)$ . There exists a measurable function  $f_\kappa = f_\kappa(x)$ ,  $x \in X$  (random variable) which may be defined for instance on the set  $X = \{x\} = [0, 1]$  equipped with Lebesgue measure  $\mathbf{P}$  such that

$$f_\kappa^*(t) = 1 - t^\kappa, \quad t \in [0, 1], \quad (2.8)$$

for example

$$f_\kappa(x) := 1 - x^\kappa, \quad x \in [0, 1];$$

then

$$f_\kappa^{**}(t) = 1 - t^\kappa/(\kappa + 1), \quad t \in [0, 1]. \quad (2.9)$$

Note that for all the values  $\kappa > 0$

$$\Theta(w) \leq \frac{\|f_\kappa\|_w}{\|f_\kappa\|_w^*} = \frac{\sup_{t \in (0,1)} [w(t)(1 - t^\kappa)]}{\sup_{t \in (0,1)} [w(t)(1 - t^\kappa/(\kappa + 1))]} = \frac{G_\kappa(w)}{H_\kappa(w)} = K_\kappa(w), \quad (2.10)$$

therefore

$$\Theta(w) \leq \inf_{\kappa \in (0, \infty)} K_\kappa(w) = K(w). \quad (2.11)$$

### 3 Examples.

We assume in this section  $w \in V$ .

**Proposition 1.** Let  $w = w(s) = w_p(s) = s^{1/p}$ ,  $p = \text{const} > 1$ ; then

$$\Theta(w_p) = 1. \quad (3.1)$$

**Proof.** The lower bound  $\Theta(w) \geq 1$ ,  $w \in V$  is obvious, see e.g. [21], chapter 5, section 3, as long as  $f^{**}(t) \geq f^*(t)$ .

In order to prove the upper estimate  $\Theta(w_p) \leq 1$  we use the assertion of theorem 1. Namely, let  $\kappa = \text{const} \in (0, \infty)$ ; note that

$$\lim_{\kappa \rightarrow \infty} \left[ \frac{f_\kappa^*(t)}{f_\kappa^{**}(t)} \right] = 1, \quad t \in (0, 1);$$

hence it is naturally to hope that

$$\lim_{\kappa \rightarrow \infty} \frac{G_\kappa(w_p)}{H_\kappa(w_p)} = 1. \quad (3.2)$$

In detail, we find by direct computations:

$$G_\kappa(w_p) = \sup_{t \in (0,1)} \left( t^{1/p} - t^{\kappa+1/p} \right) = (\kappa p + 1)^{-1/(\kappa p)} \cdot \frac{\kappa p}{\kappa p + 1}; \quad (3.3)$$

$$H_\kappa(w_p) = \sup_{t \in (0,1)} \left( t^{1/p} - t^{\kappa+1/p} / (\kappa + 1) \right) = \frac{(\kappa + 1)^{1/(\kappa p)}}{(\kappa p + 1)^{1/(\kappa p)}} \cdot \frac{\kappa p}{\kappa p + 1}; \quad (3.4)$$

$$\frac{G_\kappa(w_p)}{H_\kappa(w_p)} = (\kappa + 1)^{1/(\kappa p)}, \quad (3.5)$$

$$K(w) = \inf_{\kappa \in (0, \infty)} K_\kappa(w) \leq \lim_{\kappa \rightarrow \infty} K_\kappa(w) = \lim_{\kappa \rightarrow \infty} \frac{G_\kappa(w)}{H_\kappa(w)} =$$

$$\lim_{\kappa \rightarrow \infty} (\kappa + 1)^{1/(\kappa p)} = 1. \quad (3.6)$$

**Remark 3.1.** At the same result as in proposition 1 is true for the functions of a view

$$w_{p,q}(s) = s^{1/p} (|\log s| + 1)^{1/q}, \quad p > 1, q \in R; \quad (3.7)$$

$$w_{p,q,r}(s) = s^{1/p} (|\log s| + 1)^q (\log(|\log s| + 3))^r, \quad p > 1, q, r \in R \quad (3.8)$$

etc.

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